Near-optimal configurations in mean-field disordered systems

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(Received 10 July 2003; published 14 October 2003)

We present a general technique to compute how the energy of a configuration varies as a function of its overlap with the ground state in the case of optimization problems. Our approach is based on a generalization of the cavity method to a system interacting with its ground state. With this technique we study the random matching problem as well as the mean-field diluted spin glass. As a by-product of this approach we calculate the de Almeida-Thouless transition line of the spin glass on a fixed connectivity random graph.

DOI: 10.1103/PhysRevE.68.046706 PACS number(s): 02.70.−c, 75.10.Nr, 89.20.Ff

I. INTRODUCTION

The study of the ground state properties of disordered systems reveals deep connections with the field of random combinatorial optimization: in fact, combinatorial optimization problems can be stated in terms of zero-temperature statistical properties of some disordered system Hamiltonian. A closely related issue is the *computational complexity* of a given problem that can be stated in terms of the typical amount of time (e.g., CPU cycles) that is needed to solve the problem as a function of its size [1]. Easy/hard thresholds have been observed in different combinatorial optimization problems [2–5], and a currently highly debated issue is the interplay between the onset of the phase transition and the slowing down of local search algorithms [6]. A better understanding of the organization of the lowest energy configurations is then a promising research program not only to relate the *computational complexity* of a given problem to its statistical mechanics counterpart but also as a guide for the implementation of more efficient algorithms.

Recently, the problem of diluted mean-field spin glasses received a renewed interest by means of the *cavity approach* [7,9]. Interestingly this method turned out to be a powerful tool also to deal with combinatorial optimization problems. In this paper we study the problem of the organization of the lowest energy configurations within this framework. Our aim is to explore the way the energy of a configuration varies as a function of its overlap with the ground state. The naive procedure consisting in computing the ground state, then picking at random configurations and looking at their distance to the ground state is not computationally practicable: configurations are far too numerous for this sampling to be efficient, let alone that one has to do it for numerous instances of the coupling constants in order to get relevant averaged quantities. We need to generate configurations whose energy density is somewhat under control. The technique we use consists in perturbing the Hamiltonian to increase the energy of the ground state, then compute the new ground state. The perturbation is chosen proportional to a small parameter ε which we can tune to make the new ground state more or less distant from the original one: hereafter we will refer to this method as ε-coupling. This is actually not a new idea [10–14], however to our knowledge so far this technique has been implemented only in numerical simulations. Here we show that an analytic solution is possible in the case of the simple random matching problem as well as the case of spin glasses on random graphs of fixed connectivity.

In a recent paper Aldous and Percus [15] used similar techniques to study the matching problem (both in the Euclidean and in the mean-field versions) and the traveling salesman problem, conjecturing that it is possible to classify optimization problems into different universality classes according to the dependence of the ground state solution on small perturbations. We will compare our results to those presented in Ref. [15], and we will explain how the replica symmetry breaking (RSB) transition can be detected within the cavity scheme. Let us point out that we will restrict our analysis to the level of the replica symmetric (RS) approximation although in principle there is no problem to extend the same analysis to higher level of replica symmetry breaking; we will discuss this possibility and the potential interest of this generalization in the context of other combinatorial optimization problems (e.g., SAT [2] and coloring [3]).

The rest of the paper has the following structure. In Sec. II we study the random simple matching problem: after a description of the model we show how the cavity approach works in general for this model and we introduce a generalization in order to deal with ε-coupled systems. In Sec. III we explain how the method works in the case of the diluted spin glass where a RSB transition is known to exist. In Sec. IV some conclusions and perspectives are presented.

II. THE RANDOM SIMPLE MATCHING PROBLEM

A. The model

Given an unoriented graph $G=(V,E)$, where V are the vertices and E are the edges, a *matching* M is a set of edges

having the property that no two edges in M have an end in common. We say that a vertex $v \in V$ is *matched* if there is an edge incident to v in the matching. Otherwise the vertex is *unmatched*. A matching is called *perfect* if every vertex of G is matched.

In the following we stick to the case where G is *complete*, i.e., there is an edge between any two vertices, and when we speak of a matching we mean a perfect matching.

We introduce a function l defined on the set of edges E which associates a real number $l(e)$ with each edge $e \in E$. The quantity $l(e)$ can be thought of as a distance, or—depending on the taste—as a weight, a cost, etc. We define the total length of a matching M as $L_M = \Sigma_{e \in M} l(e)$. The matching problem consists in finding the cheapest matching, i.e., which minimizes L_M.

A classic example is the random Euclidean matching problem: the vertices are identified with N points drawn at random with the flat measure in the unit hypercube of a d-dimension Euclidean space. The cost $l(e)$ of an edge $e = \{v,w\}$ is the usual Euclidean distance between the two vertices v and w.

A mean-field approximation of the Euclidean problem has been widely investigated [16–21]: the weights of the edges are independent identically distributed (i.i.d.) random variables whose common probability distribution ρ is defined over an interval $[0,+\infty[$. The function $\rho(l)$ is assumed to behave like an integer power law for small l:

$$\rho(l) \sim \frac{l^r}{r!}. \tag{1}$$

In the thermodynamic limit $N \to \infty$ limit, the mean distance of two nearest neighbors goes to 0 like $N^{-\delta}$, where $\delta = 1/(r+1)$. Intuitively a minimum matching will only include edges of this order of magnitude, so that the only relevant feature of ρ is its behavior around 0, i.e., the r exponent. The one-edge and two-edge length distributions in this model match the ones of the Euclidean random matching problem in dimension $d = r+1$ for short distances.

Hereafter we will concentrate on the case $r=0$. We will use the following conventions: there are N vertices; the distances $l_{ij} = l_{ji}$ between two vertices are i.i.d. random variables distributed following the flat distribution over an interval $[0,N]$ [this corresponds to a rescaling of a factor N with respect to Eq. (1)]; as a shorthand notation, we indicate the set of these coupling constants as ℓ. The length of the minimum matching is an extensive quantity, and the energy (Hamiltonian) of a matching can be defined equal to its length. A matching M can be unequivocally represented by a contact matrix n_{ij} such that (1) $n_{ij} \in \{0,1\}$ and $n_{ij} = n_{ji}$; (2) $n_{ii} = 0$, no self-linkage is allowed; (3) $\forall i$, $\Sigma_{j=1}^{N} n_{ij} = 1$, each site cannot be linked more than once. Obviously $n_{ij} = 0$ if the edge $\{i,j\}$ is not in M, and $n_{ij} = 1$ if the edge $\{i,j\}$ is in M. The entry n_{ij} is called the *occupation number* of the edge $\{i,j\}$. The Hamiltonian of the matching then reads

$$L_\ell = \sum_{1 \leqslant i < j \leqslant N} n_{ij} l_{ij}, \tag{2}$$

where ℓ plays the role of the quenched disorder and the n_{ij} are the dynamic local variables.

B. The cavity equations

The cavity equations at finite temperature for the matching problem have been derived in Ref. [21] (a comprehensive introduction to this subject is found in Ref. [22]). Let us briefly reproduce the basic steps following Ref. [23]. The partition function for the matching problem is

$$Z = \sum_{\{n_{ij}\}} \exp\left(-\beta N \sum_{1 \leqslant i < j \leqslant N} n_{ij} l_{ij} \right), \tag{3}$$

where the scaling factor N makes the free energy $F = -\ln(Z)/\beta$ an extensive quantity. Following polymer theory [25], one first introduces a more tractable representation of the partition function, which consists in mapping the matching problem onto a system of interacting spins. On each vertex i one puts a p-dimensional vector spin $\mathbf{S}_i = (S_i^1, \ldots, S_i^p)$ normalized by $\mathbf{S}_i^2 = p$. Let $d\mu$ be the integration measure on the corresponding sphere. If we define the coupling constants $T_{ij} = \exp(-\beta N l_{ij})$, one can check that the partition function (3) can be written as

$$Z = \lim_{p \to 0} \int \left[\prod_{i=1}^{N} S_i^1 d\mu(\mathbf{S}_i) \right] \exp\left(\sum_{1 \leqslant i < j \leqslant N} T_{ij} \mathbf{S}_i \cdot \mathbf{S}_j \right). \tag{4}$$

Expanding the exponential into a power series and applying the following property:

$$\lim_{p \to 0} \int d\mu(\mathbf{S}_i) S_i^{\alpha_1} S_i^{\alpha_2} \cdots S_i^{\alpha_q} = \delta_{q,2} \delta_{\alpha_1, \alpha_2}, \tag{5}$$

one can easily recover Eq. (3) from Eq. (4). Note also that the magnetization vector of spin i has components $m_i^\alpha = \delta_{\alpha,1} m_i$.

The cavity method consists in adding a new spin $\mathbf{S}_0$ to an N-site system $\{\mathbf{S}_1, \ldots, \mathbf{S}_N\}$. The partition function is calculated assuming that the statistical correlations in the N-site system can be neglected. More rigorously we make use of the *clustering theorem*, tacitly assuming that the system has just one pure state. We can thus encode the effect of the whole system onto each spin i as an effective field h_i. The $N+1$-site partition function can be written as

$$Z_{N+1} = \lim_{p \to 0} \int \left[\prod_{i=1}^{N} S_i^1 d\mu(\mathbf{S}_i) \right] \exp\left(\sum_{i=1}^{N} h_i S_i + \sum_{i=1}^{N} T_{0i} \mathbf{S}_0 \cdot \mathbf{S}_i \right), \tag{6}$$

and can be easily computed thanks to Eq. (5), giving

$$m_0 = \left(\sum_{i=1}^{N} T_{0i} m_i \right)^{-1}, \tag{7}$$

where we have used the fact that m_i, the magnetization of site i before the addition of site 0, is $1/h_i$. At the end of the day we are not interested in the spin variables but in the solution of the matching problem. The thermal average of the occupation number of the edge $0-i$ is simply related to the magnetizations by

$$\langle n_{0i}\rangle = m_0 T_{0i} m_i^c . \tag{8}$$

Since we will be interested eventually in the ground state properties of this model we have to take the $\beta\to\infty$ limit of our equations. Following Ref. [27] it is useful to set

$$m_i = e^{\beta\phi_i} \quad \text{for} \quad i\in\{1,\ldots,N\}. \tag{9}$$

The zero-temperature limit of Eqs. (7) and (8), thanks to Eq. (9), reduces to the following zero-temperature cavity relation:

$$\phi_0 = \min_{i=1,\ldots,N} N l_{0i} - \phi_i , \tag{10}$$

$$n_{0i} = \delta_{i,i*}, \tag{11}$$

where $i*$ is the index attaining the minimum $N l_{0i} - \phi_i$.

C. The ε-coupling method

The idea of the ε-coupling method in the context of the matching problem is the following: given the set ℓ of l_{ij} distances (called 0 distances), one first finds the minimum matching (called 0–ground state), which is characterized by some occupation numbers n_{ij}. Then one perturbs the lengths of the edges of the graph by adding a quantity ε to the edges present in the 0–ground state. Formally the edge lengths become the following ε distances:

$$l_{ij}^{\varepsilon} = l_{ij} + \varepsilon n_{ij} . \tag{12}$$

One solves the matching problem with these ε distances and obtains a solution we will name ε–ground state, which is expected to be different from the 0–ground state. The larger the ε, the stronger the 0–ground state is penalized. The ε–ground state is characterized by the occupation numbers n_{ij}^{ε}. Two quantities are of interest: first the difference of length (energy) between the ε–ground state and the 0–ground state computed with the 0 distances:

$$\Delta L_\ell = \sum_{i<j} (n_{ij}^{\varepsilon} - n_{ij}) l_{ij} . \tag{13}$$

Second is the distance d between the ε–ground state and the 0–ground state:

$$d_\ell = 1 - q_\ell , \tag{14}$$

where the overlap q_ℓ is equal to the proportion of edges in common:

$$q_\ell = \frac{2}{N} \sum_{i<j} n_{ij}^{\varepsilon} n_{ij} . \tag{15}$$

More precisely one would like to compute the average with respect to the coupling constants of these quantities, so let us define $\Delta L = \overline{\Delta L_\ell}$ and $q = \overline{q_\ell}$. An analytic approach is possible in the framework of the cavity method. We have two spin systems, the 0 system and the ε system, standing on the same graph, and which are coupled. More precisely the Hamiltonian of the ε system is conditioned by the ground state of the 0 system.

While the 0 system certainly obeys the cavity equations (10), how to deal with the ε system is more problematic. A *naive* approach would lead us to

$$\tau_0 = \min_{i=1,\ldots,N} \gamma_i , \tag{16}$$

where

$$\gamma_i = l_{0i} - \tau_i \quad \text{if} \quad i \neq i*, \tag{17}$$

$$\gamma_i = l_{0i} + \epsilon - \tau_i \quad \text{if} \quad i = i*. \tag{18}$$

Nevertheless these equations are wrong. They would be true if the interactions between the old spins in the ε system of $N+1$ spins were the same as in the ε system of N spins, and this is not exactly the case: when adding the new spin to the 0 system, it gets matched to one of the old spins, whose previous match becomes unmatched. To circumvent this difficulty, one should distinguish between matched and unmatched spins in the 0 system. The variable τ_i of the ε system will be called f_i if the vertex i is matched in the 0 system, and v_i if it is not. The correct equations are thus

$$v_0 = \min_{i=1,\ldots,N} (l_{0i} - f_i), \tag{19}$$

$$f_0 = \min_{i=1,\ldots,N} \gamma_i , \tag{20}$$

where

$$\gamma_i = l_{0i} - f_i \quad \text{if} \quad i \neq i*, \tag{21}$$

$$\gamma_i = l_{0i} + \epsilon - v_i \quad \text{if} \quad i = i*. \tag{22}$$

The new spin in the ε system gets matched to the spin $i**$, the index which minimizes Eq. (20). The contribution to the overlap q is $\delta_{i*,i**}$. The contribution to ΔL is $l_{0i**} - l_{0i*}$.

When averaging over the disorder, in the thermodynamic limit the quantities (φ_0, v_0, f_0) and (φ_i, v_i, f_i) are i.i.d. random variables (beware that φ, v, and f on the same site are correlated). The above Eqs. (10) and (19) define a stochastic flow whose fixed point is the limit distribution of (φ, v, f). We use a population algorithm similar to the one discussed in Refs. [7,9] to solve the equations. In order to save computing time, we use a fluctuating connectivity approximation of the matching problem: we keep the only edges whose lengths are smaller than a given cutoff z, so that the connectivity of a vertex is a Poisson random variable of mean z. We store a large population of $\mathcal{N}$ triplets (φ_i, v_i, f_i), $i=1,\ldots,\mathcal{N}$, which we initialize randomly and update iteratively: at each step an integer k is extracted following the Poisson distribu-

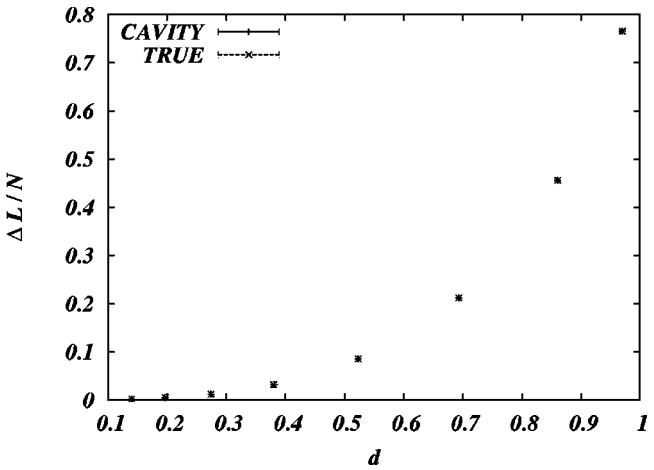

FIG. 1. $\Delta L/N$ vs d. The points obtained by the cavity population algorithm with $z=30$, $\mathcal{N}=200\,000$, and 10^8 iterations (CAVITY) coincide with the measures obtained by averaging over 4000 samples of total number of points $N=400$ (TRUE).

tion of mean z, k random elements of the population are chosen and we compute (φ_0, v_0, f_0) following the scheme defined by Eqs. (10), (19), and (20) limiting the search for the min to the k extracted triplets; the resulting (φ_0, v_0, f_0) overwrites an element of the population chosen at random. Once stochastic convergence of the population is achieved, we keep on iterating and compute *en passant* the contributions to ΔL and q. Their flat averages over many steps provide $\overline{\Delta L}$ and $\overline{d}$. The output of this algorithm is presented in Figs. 1 and 2. Beside we have calculated the ground state of 4000 matching instances using BLOSSOM 4 software [26] and we have tested the results against the cavity approach. In Fig. 1 we display $\Delta L/N$ vs d: the cavity approach is in perfect agreement with the direct calculation. In Fig. 2 we display the $\Delta L/N$ vs d curve (main panel) and the best one-parameter fit of the form const$\times d^3$, while in the inset we display the d vs ϵ curve together with the best one-parameter

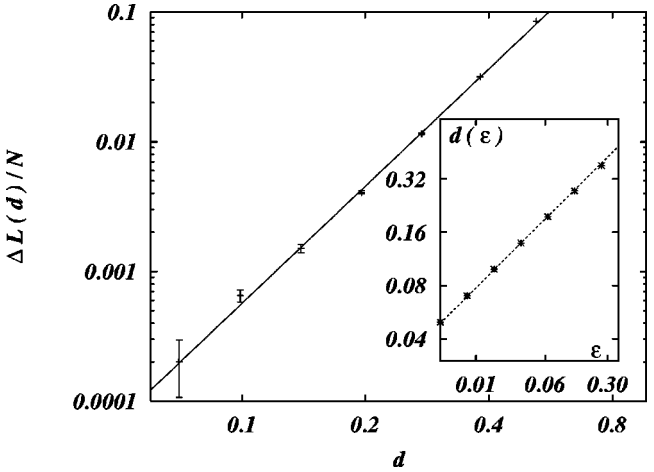

FIG. 2. Main panel: Average length $\Delta L/N$ vs distance d, continuous line is the best fit of the form const$\times d^3$. Inset: Average distance d vs coupling parameter ϵ; dotted line is the best fit of the form const$\times\sqrt{\epsilon}$.

fit of the form const$\times\sqrt{\epsilon}$: in both cases the reduced χ^2 is of the order of 1. Note that a simple scaling argument shows that the two exponents are not independent: let us assume that for ϵ small enough $d\sim\epsilon^\beta$, then from Eq. (13) $\Delta L/N \sim \epsilon^{\beta+1}\sim d^{(\beta+1)/\beta}$. The scaling exponent α introduced by Aldous and Percus [15] is easily recovered setting $\alpha=(\beta+1)/\beta$. We find that the random matching features a scaling exponent $\beta=1/2$ ($\alpha=3$), in agreement with the results found in Ref. [15]. A completely analytic study of the coupled system of Eqs. (19) and (20) might be done, but we did not undertake it.

An important remark is in order: in Ref. [24] Aldous introduces and proves the following *asymptotic essential uniqueness* property:

Let M be the generic element of a family of matchings depending both on N and on the realization of the l_{ij}; we call $q_{M,min}$ the overlap between M and the minimum matching, and $d_{M,min}=1-q_{M,min}$ their distance.

For each $0<\delta<1$ there exists $\varepsilon(\delta)>0$ such that: if $\forall N$, $d_{M,min}\geqslant\delta$ then

$$\liminf_N \frac{L_M}{N} \geqslant \frac{\pi^2}{6}+\varepsilon(\delta). \tag{23}$$

In physical terms: in the thermodynamic limit, a matching (configuration) which differs from the ground state by a nonzero proportion of edges has an energy density strictly greater than that of the ground state. The other way round: a state with the same intensive energy density as the ground state can be obtained only by changing a nonextensive number of edges in the ground state. This is the proof that there is no RSB, at least at zero temperature. The plot in Fig. 1 can be seen as an illustration of this theorem.

To conclude, note that our perturbation of the energy is $O(N)$ so the information we get on the energy landscape is limited. In particular we do not explore the lowest lying excited configurations which have been numerically shown to have an energy $\Delta L\sim 1/\sqrt{N}$ and $d\sim 1/\sqrt{N}$ [28]. This is a limitation of our analytic approach: it is purely thermodynamic so that we cannot have a hint at finite size [29] effects.

III. THE GAUSSIAN SPIN GLASS ON THE BETHE LATTICE

In close analogy with the preceding section, we derive cavity equations for the ε-coupling method applied to the Gaussian spin glass on the Bethe lattice of connectivity $k+1$, in the presence of an external field. We keep to the level of the RS approximation. We will see that our results provide a self-consistency check of this hypothesis which enables us to trace out when it is valid or not.

Throughout this section we make thorough use of the notations introduced in our previous paper [8]. We refer to Secs. II and III therein for details.

A. The cavity equations for a single system at zero temperature

First, following Mézard and Parisi [9] we work out the cavity method directly at zero temperature for a single sys-

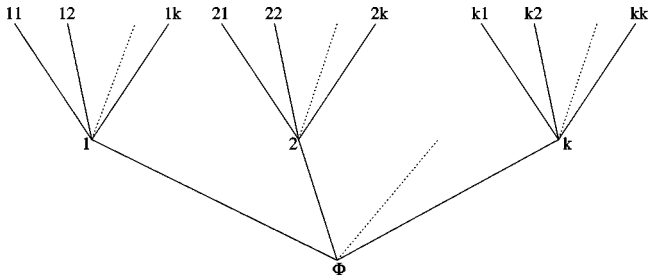

FIG. 3. A branch of a Cayley tree of connectivity k.

tem of N spins σ interacting through coupling constants J—their set being denoted $\mathcal{J}$—which are i.i.d. random variables with a Gaussian distribution of mean 0 and variance 1, in the presence of an external field h_{ext}.

Let us consider the merging process of k branches rooted at spins $i=1,\ldots,k$ onto a new spin Φ illustrated in Fig. 3. We look at how the energy of the ground state evolves under this process. Before the merging, on a given branch, we let the spin at the root undetermined. Thus the ground state energy of the branch rooted at spin i is conditioned by the value of the spin i and can be written

$$E(\sigma_i)=A_i-h_i\sigma_i, \tag{24}$$

where A_i is a constant and h_i is an effective field (beware that it is not the local field). Note that h_i contains the effect of the external field h_{ext}. As a consequence, the energy of the system of the k branches before the merging is conditioned on the values of the spins $1,\ldots,k$, and reads

$$E(\sigma_1,\ldots,\sigma_k)=A_1+\cdots+A_k-h_1\sigma_1-\cdots-h_k\sigma_k. \tag{25}$$

The system after the merging has an energy

$$E'(\sigma_1,\ldots,\sigma_k,\sigma_\Phi)=A_1+\cdots+A_k-(h_1+J_{\Phi,1}\sigma_\Phi)\sigma_1 - \cdots-(h_k+J_{\Phi,k}\sigma_\Phi)\sigma_k-h_{ext}\sigma_\Phi, \tag{26}$$

which, in order to be that of the ground state, is to be minimized with respect to $\sigma_1,\ldots,\sigma_k$ at fixed σ_Φ. This is realized by independently choosing the sign of each σ_i such that $(h_i+J_{\Phi,i}\sigma_\Phi)\sigma_i=|h_i+J_{\Phi,i}\sigma_\Phi|$. As we can write

$$|h_i+J_{\Phi,i}\sigma_\Phi|=\omega(h_i,J_{\Phi,i})+\lambda(h_i,J_{\Phi,i})\sigma_\Phi, \tag{27}$$

where

$$\omega(h,J)=\frac{|h+J|+|h-J|}{2},$$

$$\lambda(h,J)=\frac{|h+J|-|h-J|}{2}, \tag{28}$$

we get the appealing following form:

$$E'(\sigma_\Phi)=A_1+\cdots+A_k-\sum_{i=1}^{k}\omega(h_i,J_{\Phi,i}) - \left(\sum_{i=1}^{k}\lambda(h_i,J_{\Phi,i})+h_{ext}\right)\sigma_\Phi. \tag{29}$$

By comparison with Eq. (24) this leads to the following recursion relation:

$$h_\Phi=\sum_{i=1}^{k}\lambda(h_i,J_{\Phi,i})+h_{ext}. \tag{30}$$

B. The cavity equations for two coupled systems

Now we imagine there are two spin systems sitting on the N vertices A of the *same* Bethe lattice.

(1) The σ spins obey the same Hamiltonian as in Sec. III A:

$$H[\sigma]=-\sum_{\langle A,B\rangle}J_{A,B}\sigma_A\sigma_B-h_{ext}\sum_A\sigma_A. \tag{31}$$

Let us call $\sigma^*=(\sigma_A^*)$ the ground state of this Hamiltonian.

(2) The τ spins obey the following perturbed Hamiltonian, conditioned on the σ_A^*:

$$H[\tau|\sigma^*]=-\sum_{A,B}J_{A,B}\tau_A\tau_B-h_{ext}\sum_A\tau_A-\epsilon\sum_A\sigma_A^*\tau_A. \tag{32}$$

The choice $\varepsilon>0$ corresponds to an attractive interaction for the τ spin variables to the configuration σ^*, the choice $\varepsilon<0$ to a repulsion. The ground state of Hamiltonian (32) is named $\tau^*=(\tau_A^*)$.

We are interested in the overlap

$$q\equiv\overline{q_{\mathcal{J}}}=\frac{1}{N}\sum_i\overline{\sigma_i^*\tau_i^*}, \tag{33}$$

where with the symbol $\overline{O_{\mathcal{J}}}$ we mean the average of a generic $\mathcal{J}$-dependent observable $O_{\mathcal{J}}$ over the different sample realizations. The correlations between the σ spins and the τ spins can be accounted for by stating that the value of the effective field acting on spin τ_Φ depends on the value of spin σ_Φ, where Φ is the root of an isolated branch. So we have to store three quantities related to the root Φ of a branch: (1) h_Φ, the effective field acting on spin σ_Φ; (2) h_Φ^+, the effective field acting on spin τ_Φ under the condition that spin σ_Φ has the value $+1$; (3) h_Φ^-, the effective field acting on spin τ_Φ under the condition that spin σ_Φ has the value -1.

1. Iteration

When performing the merging of k branches onto a new site, the equation for the system of the σ spins is exactly the same as in Sec. III A:

$$h_\Phi = \sum_{i=1}^{k} \lambda(h_i, J_{\Phi,i}) + h_{ext}. \tag{34}$$

As far as the τ spins are concerned, one must compute h_Φ^+ as well as h_Φ^-. So we assume in turn that $\sigma_\Phi = +1$ and $\sigma_\Phi = -1$. This determines the values of the σ_i^*:

$$\sigma_i^* = \mathrm{sgn}(h_i + J_{\Phi,i}\sigma_\Phi^*), \tag{35}$$

and the effective field acting on spin τ_Φ reads

$$h_\Phi^{\sigma_0^*} = \sum_{i=1}^{k} \lambda(h_i^{\sigma_i^*}, J_i) + \varepsilon\sigma_\Phi^* + h_{ext}. \tag{36}$$

2. Measures

To measure the overlap, one uses the merging procedure of $(k+1)$ branches onto a new vertex Ψ: the effective field acting onto spin σ_Ψ is

$$H_\Psi = \sum_{i=1}^{k+1} \lambda(h_i, J_{\Psi,i}) + h_{ext}. \tag{37}$$

By contrast with the iteration, spin σ_Ψ has a determined value, the one which minimizes the energy: $\sigma_\Psi^* = \mathrm{sgn}(H_\Psi)$. This also determines the values of the σ_i^* according to Eq. (35). Eventually one can compute the effective field acting on spin τ_Ψ:

$$H_\Psi = \sum_{i=1}^{k+1} \lambda(h_i^{\sigma_i^*}, J_i) + \varepsilon\sigma_0^* + h_{ext}, \tag{38}$$

whose sgn gives τ_Ψ^*. The contribution to the overlap is $\sigma_\Psi^*\tau_\Psi^*$.

C. The results

Unless otherwise stated, the following results are for $k=2$. Once again they have been obtained by a population algorithm. Here the population is made of $\mathcal{N}$ triplets (h, h^+, h^-).

In the case $\varepsilon > 0$ we expect q to be equal to 1, regardless of the value of h_{ext}: the ground state of the perturbed Hamiltonian (32) should be the same as Hamiltonian (31). We will see that so it is. Thus the interesting regime is $\varepsilon \leqslant 0$.

What we obtain in the case $h_{ext}=0$ is plotted in Fig. 4. The fact that $q=-1$ for every $\varepsilon<0$ is due to the symmetry of the original Hamiltonian (31) under reversal of all the spins: the ground state of the perturbed Hamiltonian is $\tau_A^* = \sigma_A^*$. More interesting is the fact that at $\varepsilon=0$, $q=0$: assuming RS one would expect $q=1$. This is a sign that the RS ansatz is not self-consistent, and is to be dismissed.

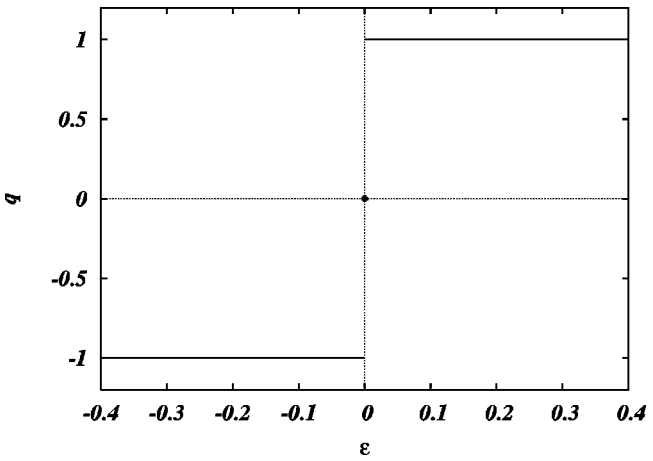

FIG. 4. q vs ε for $k=2$, in the case $h_{ext}=0$.

Things may be more convincing if one turns the external field on. The case $h_{ext}=0.1$ is plotted in Fig. 5. As the symmetry of the original Hamiltonian is lifted, the plot of q as a function of $\varepsilon<0$ is no longer a constant equal to -1. It does, however, tend to -1 when $\varepsilon \to -\infty$ because in this limit the attracting term between the σ system and the τ system dominates the perturbed Hamiltonian (32). When $\varepsilon \to 0^-$, q goes continuously to a value which is no longer 0, but is still not 1. Again the RS ansatz has problems.

We do, however, expect the RS ansatz to be valid for a sufficiently high external magnetic field. Fixing $\varepsilon=0$, we let h_{ext} increase. See Fig. 6: it appears that q is an increasing function of h_{ext} and it saturates to 1 at $h_{ext}^c \sim 0.48$. It is the sign that the RS ansatz becomes self-consistent above h_{ext}^c.

For $k=5$ we find $h_{ext}^c \sim 1.86$ which is different from the value $h_{ext}^c \sim 2.1$ found in Ref. [30] by the analysis of numerical simulations. We believe our result is exact, and the discrepancy can be explained by the fact that their result relies on finite size scaling arguments with relatively poor precision.

What we got here is actually the point at $T=0$ of the de Almeida-Thouless (AT) line [31] for the Gaussian spin glass on the Bethe lattice. We had the idea that we could

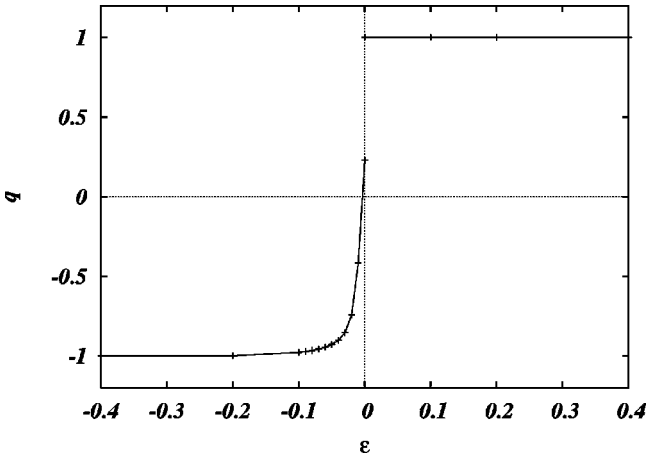

FIG. 5. q vs ε for $k=2$, in the case $h_{ext}=0.1$, obtained with $\mathcal{N}=2000$ and $100\,000\mathcal{N}$ iterations.

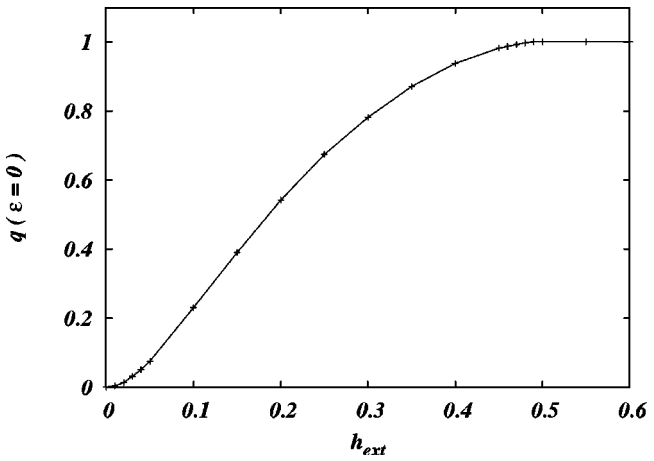

FIG. 6. q vs h_{ext} for $k=2$, $\varepsilon=0$, obtained with $\mathcal{N}=2000$ and $100\,000\mathcal{N}$ iterations.

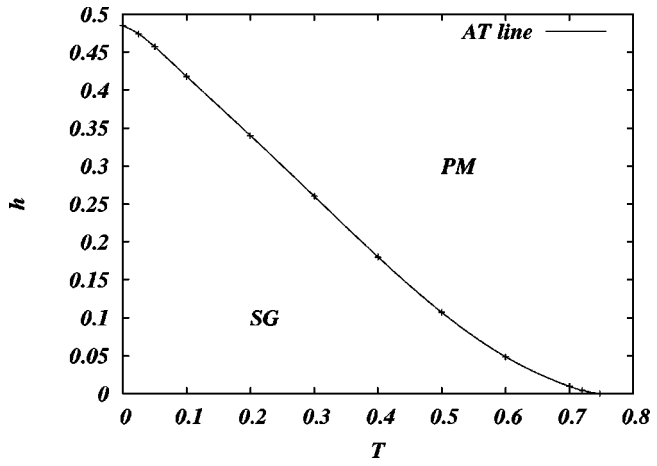

FIG. 7. The de Almeida-Thouless line (h vs T) on the Bethe lattice in the case $k=2$, separating spin-glass (SG) phase from the paramagnetic (PM) phase. Data are obtained with $\mathcal{N}=10\,000$ and $10\,000\mathcal{N}$ iterations. We estimate error bars to be hardly visible on this scale. Continuous line is just a guide to the eye.

generalize our approach to a nonzero temperature to determine the whole line.

D. Searching for the AT line

It is not difficult to generalize the argument of Sec. III B (in the only case $\varepsilon=0$) to a nonzero temperature. We consider two noninteracting systems, the σ spins and the τ spins, standing on the *same* Bethe lattice and obeying the *same* Hamiltonian (31). Solving this double system at the level of RS is easy: we use the population algorithm of Sec. III A adapted to follow simultaneously two populations. A crucial point is that whenever one randomly extracts sites or coupling constants, they are the *same* for the σ population and the τ population. This procedure enables us to measure the average overlap between the two systems:

$$q \equiv \overline{q_{\mathcal{J}}} = \overline{\langle \sigma_A \rangle \langle \tau_A \rangle}. \tag{39}$$

The criterion for the RS ansatz to be self-consistent is

$$q = \overline{m_A^2}, \tag{40}$$

where m_A is the local magnetization measured either in the σ system or in the τ system (they are obviously equal).

Given a value of T, we run the algorithm for increasing values of h_{ext}, so as to determine its value $h_{ext}^c(T)$ beyond which condition (40) holds. The plot $h_{ext}^c(T)$ is the AT line, see Fig. 7. To our knowledge it is the first time the AT line has been obtained for a spin glass on the Bethe lattice. Two predictions made in Ref. [32] can be checked. First, the critical temperature T_c, such that $h_{ext}^c(T_c)=0$, is the solution of the equation $k\overline{\tanh^2(J/T_c)}=1$: in the case $k=2$ this gives $T_c=0.748$. Second, close to T_c, $h_{ext}^c(T)$ should behave like $(T_c-T)^{3/2}$, which a numerical fit of our data confirms.

IV. CONCLUSIONS AND PERSPECTIVES

In this paper we presented the derivation and implementation of the ε-coupling method in the framework of the cavity approach. This technique allows us to explore the way the energy of a configuration varies as a function of its overlap with the ground state and more generally to address the problem of the organization of the lowest energy configurations. The lesson we get applying this method to the case of the simple random matching problem is very clear: the space of the lowest energy configurations is organized such that their energy difference with respect to their distance from the ground state scales as $\Delta E/N \propto d^3$. A situation like this, or in general whenever $\Delta E/N \propto d^\alpha$ with $\alpha>0$, is related to the property of replica symmetry of the system, which implies Aldous *asymptotic essential uniqueness* property [24].

A similar study presented in Ref. [15] suggests that this property is also shared by the minimum spanning tree problem, the minimum matching problem in Euclidean dimension $d=1$, and the traveling salesman problem also in Euclidean dimension $d=1$ (all with $\alpha=2$). Minimum matching problem and traveling salesman problem in $d=2,3$ are instead characterized by $\alpha=3$ as the mean-field matching problem we have studied.

A simple case with no *asymptotic essential uniqueness* property is the spin glass on a fixed connectivity random graph studied in Sec. III. Indeed our computation based on the RS assumption yields that $\lim_{\epsilon\to 0^-} d \neq 0$ which is a physical nonsense since the model has a unique ground state (the couplings are Gaussian). This inconsistency tells us not only that—as we already know after [7,9]—the cavity approximation must be improved in order to take into account the presence of many states but also gives us a practical tool to probe the phase space for the onset of full RSB: the search for the AT line in the case of the spin glass on a fixed connectivity random graph is a simple and instructive example.

A very interesting issue is the generalization of the ε-coupling method to the case where the *asymptotic essential uniqueness* does not hold. In the last year a compact and efficient formalism has been developed to apply the cavity method to SAT and coloring problems [2,3], at the level of

1RSB, evidentiating a clustering transition. This clustering transition consists in the sudden appearance of an exponential number of metastable states, which—intuitively—cause local search algorithm to get stuck. We believe that it is possible to generalize the ε-coupling method presented in this paper to the 1RSB level, although at a higher computational cost. This could give interesting results on the inner mechanism of the clustering of states. Open problems, such as how large a single cluster could be and what the inter-cluster mean distance is, could be addressed within this formalism.

ACKNOWLEDGMENTS

We are grateful to F. Ricci-Tersenghi for many interesting discussions. A.P. acknowledges the financial support provided through the European Community's Human Potential Program under Contract No. HPRN-CT-2002-00307, DYGLAGEMEM.

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